Isomorphisms within Hopf-Galois structures on separable field extensions

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6. Looking forward

Let L/K be a finite separable extension of fields, normal closure E.

Let $G = \mathrm{Gal}(E/K), G' = \mathrm{Gal}(E/L), X = G/G'$ (left cosets).

Thanks to Greither-Pareigis, later Childs-Byott, Hopf-Galois structures on L/K can be classified using two techniques:

1. Regular subgroups $N \leq \mathrm{Perm}(X)$ normalized by G, where $g \in G$ acts on N via conjugation by $\lambda(g) \in \mathrm{Perm}(X)$.
2. Embeddings $\beta : G \hookrightarrow \mathrm{Hol}(N) = \rho(N) \cdot \mathrm{Aut}(N) \subset \mathrm{Perm}(N)$ such that $\beta(G')$ is the stabilizer of 1_N. Note β_1 and β_2 induce the same Hopf-Galois structure if and only if $\beta_1(g) = \theta\beta_2(g)\theta^{-1}$ for some fixed $\theta \in \mathrm{Aut}(N)$. ($N$ an abstract group of order $|G|$.)

The corresponding Hopf algebras are of the form $E[N]^G$, where $g \in G$ acts on E by the Galois action, and on N by either conjugation by $\lambda(g)$ (technique 1) or via the image $\overline{\beta}$ of β projected onto $\mathrm{Aut}(N)$ (technique 2).

The problems

Suppose two Hopf-Galois structures are found with underlying Hopf algebras H_1, H_2, corresponding to two (possibly isomorphic) groups N_1, N_2, or two embeddings β_1, β_2.

1. Under what conditions are $H_1 \cong H_2$ as K-Hopf algebras?
2. Does there exist a field $K \subseteq F \subseteq E$ such that $F \otimes H_1 \cong F \otimes H_2$ as F-Hopf algebras?
3. Under what conditions are $H_1 \cong H_2$ as K-algebras?
4. Does there exist a field $K \subseteq F \subseteq E$ such that $F \otimes H_1 \cong F \otimes H_2$ as F-algebras?

Furthermore, how many of these questions can be solved using group theory alone?

Can this happen at all?

Yes [Childs, 2013].

Let L/K be Galois, group G.

Let $\psi : G \to G$ be a homomorphism such that $\psi(gh) = \psi(hg)$ for all $g, h \in G$ and $\psi(g) = g$ iff $g = 1_G$.

Let $N_\psi = \{\lambda(g)\rho(\psi(g)) : g \in G\} \leq \mathrm{Perm}(G)$.

Then the Hopf algebra $L[N_\psi]^G$ is isomorphic to H_λ, the Hopf algebra which gives the canonical nonclassical Hopf Galois structure, corresponding to $N = \lambda(G)$.

Inseparable analogue?

Suppose (for this slide only) that L/K is purely inseparable.
Then Hopf algebra isomorphisms are common, algebra isomorphisms even more so.

Example

Suppose $\mathrm{char}(K) = p > 2$ and $[L : K] = p$. Then L/K admits an infinite number of Hopf-Galois structures, each with underlying Hopf algebra $H = K[t]/(t^p), \Delta(t) = t \otimes 1 + 1 \otimes t$.

More generally, for $[L : K] = p^n$, any Hopf algebra which gives a Hopf-Galois extension is a truncated polynomial algebra of dimension p^n, allowing for very few isomorphism classes of K-algebras.
For example, if $[L : K] = p^4$ the only possible H are:

$$H \cong K[t]/(t^{p^4}) \qquad H \cong K[t_1, t_2]/(t_1^{p^3}, t_2^p) \qquad H \cong K[t_1, t_2]/(t_i^{p^2})$$

$$H \cong K[t_1, t_2, t_3]/(t_1^{p^2}, t_2^p, t_3^p) \qquad H \cong K[t_1, t_2, t_3, t_4]/(t_i^p)$$

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The answer to #1

Keep notation from before: $K \subseteq L \subseteq E$, $G = \mathrm{Gal}(E/K)$, $G' = \mathrm{Gal}(E/L)$, $X = G/G'$.

Let $N_1, N_2 \leq \mathrm{Perm}(X)$ be regular subgroups normalized by G.

Let $H_i = E[N_i]^G, \ i = 1, 2$.

Proposition

$H_1 \cong H_2$ as K-Hopf algebras if and only if there exists a G-equivariant isomorphism $\phi : N_1 \to N_2$.

Well-known; often proved using algebraic geometry.

Proposition

$H_1 \cong H_2$ as K-Hopf algebras if and only if there exists a G-equivariant isomorphism $\phi : N_1 \to N_2$.

Non-Geometric Proof (sketch).

$\phi : N_1 \to N_2$ induces an E-Hopf algebra map $\phi : E[N_1] \to E[N_2]$ which is also G-equivariant.
If $h \in E[N_1]$ is fixed by $g \in G$, then

$$ {}^{g}\phi(h) = \phi({}^{g}h) = \phi(h), $$

so $\phi(h) \in E[N_2]$ is also fixed by G. Thus, ϕ descends to a map $H_1 \to H_2$.
Conversely, a K-Hopf algebra isomorphism $E[N_1]^G \to E[N_2]^G$ induces an E-Hopf algebra map $E[N_1] \to E[N_2]$ which is G-equivariant.
Restricting to group-likes gives the required G-invariant isomorphism $N_1 \to N_2$. □

The answer to #2

Proposition

$H_1 \cong H_2$ as K-Hopf algebras if and only if there exists a G-equivariant isomorphism $\phi : N_1 \to N_2$.

Let $K \subseteq F \subseteq E$, and let $G_0 = \mathrm{Gal}(E/F)$.

The same proof, *mutatis mutandis*, of the previous result gives us:

Corollary (TARP, 2018)

$F \otimes H_1 \cong F \otimes H_2$ as F-Hopf algebras if and only if there exists a G_0-equivariant isomorphism $\phi : N_1 \to N_2$.

Some consequences

Corollary (TARP, 2018)

$F \otimes H_1 \cong F \otimes H_2$ as F-Hopf algebras if and only if there exists a G_0-equivariant isomorphism $\phi : N_1 \to N_2$.

1. Suppose L/K Galois, and let ψ be a fixed-point free endomorphism of G. Then $N = N_\psi = \{\lambda(g)\rho(\psi(g)) : g \in G\}$. The map $\varphi : \lambda(G) \to N_\psi$ given by $\varphi(g) = \lambda(g)\rho(\psi(g))$ is readily seen to be G-equivariant, so their corresponding Hopf algebras are isomorphic over K.
2. Suppose $N_1 \cong N_2$ as groups, and let $F = E$. Then G_0 is trivial, hence any isomorphism $N_1 \to N_2$ is G_0-invariant. Thus $E \otimes H_1 \cong E \otimes H_2$, obvious since H_1 and H_2 are K-forms of isomorphic group rings.
3. If $N_1 \not\cong N_2$, no such field F exists.

Some more consequences

Corollary (TARP, 2018)

$F \otimes H_1 \cong F \otimes H_2$ as F-Hopf algebras if and only if there exists a G_0-equivariant isomorphism $\phi : N_1 \to N_2$.

4. Suppose L/K is Galois, let $Z = Z(G)$ and let $F = L^Z$. We have the classical structure $K[G]$ and the canonical non-classical structure $H_\lambda = H[\lambda(G)]^G$. The corresponding regular subgroups are $\rho(G)$ and $\lambda(G)$ respectively, and the map $\rho(g) \mapsto \lambda(g)$ is Z-invariant. Hence $F \otimes H_\lambda \cong F \otimes K[G] \cong F[G]$.

5. Suppose L/K is Galois. No nonclassical Hopf-Galois structure uses a Hopf algebra isomorphic to $K[G]$.

Byott's translation

Let $\beta_1, \beta_2 : G \to \mathrm{Hol}(N)$ be two nonequivalent embeddings, giving rise to Hopf algebras H_1, H_2 respectively.

Proposition (TARP, 2018)

$H_1 \cong H_2$ as K-Hopf algebras if and only if there exists a $\theta \in \mathrm{Aut}(N)$ such that

$$\overline{\beta_2}(g) = \theta\overline{\beta_1}(g)\theta^{-1} \text{ for all } g \in G.$$

Corollary

Let $K \subseteq F \subseteq E,\ G_0 = \mathrm{Gal}(E/F)$ Then $F \otimes H_1 \cong F \otimes H_2$ as K-Hopf algebras if and only if there exists a $\theta \in \mathrm{Aut}(N)$ such that

$$\overline{\beta_2}(g) = \theta\overline{\beta_1}(g)\theta^{-1} \text{ for all } g \in G_0.$$

An example

Let $L = K(\alpha_1, \alpha_2)$, $\alpha_1^2 = a \in K,\ \alpha_2^2 = b \in K,\ \alpha_1, \alpha_2 \notin K$.

$G = \mathrm{Gal}(L/K) = \langle g, h \rangle \cong C_2 \times C_2$.

Let $N = \langle \eta \rangle \cong C_4$. Then $\mathrm{Hol}(N) = \langle \rho(\eta) \rangle \cdot \langle \theta \rangle$ where $\theta(\eta) = \eta^3$.

$\mathrm{Hol}(N) \cong D_4$, the dihedral group of order 8, and hence has two subgroups isomorphic to G:

$$G_1 = \{1_{\mathrm{Hol}(N)}, \rho(\eta)\theta, \rho(\eta^2), \rho(\eta^3)\theta\},\ G_2 = \{1_{\mathrm{Hol}(N)}, \rho(\eta^2), \theta, \rho(\eta^2)\theta\}.$$

Thus, $\beta(G) = G_1$ or $\beta(G) = G_2$. But $\theta \in G_2$ stabilizes 1_N, so we cannot have $\beta(G) = G_2$.

Therefore, $\beta(G) = G_1$, giving six different choices for β.

$\beta(G) = \{1_{\mathrm{Hol}(N)}, \rho(\eta)\theta, \rho(\eta^2), \rho(\eta^3)\theta\}$

	1_G	g	h	gh
β_1	$1_{\mathrm{Hol}(N)}$	$\rho(\eta)\theta$	$\rho(\eta^3)\theta$	$\rho(\eta^2)$
β_2	$1_{\mathrm{Hol}(N)}$	$\rho(\eta^3)\theta$	$\rho(\eta)\theta$	$\rho(\eta^2)$
β_3	$1_{\mathrm{Hol}(N)}$	$\rho(\eta)\theta$	$\rho(\eta^2)$	$\rho(\eta^3)\theta$
β_4	$1_{\mathrm{Hol}(N)}$	$\rho(\eta^3)\theta$	$\rho(\eta^2)$	$\rho(\eta)\theta$
β_5	$1_{\mathrm{Hol}(N)}$	$\rho(\eta^2)$	$\rho(\eta)\theta$	$\rho(\eta^3)\theta$
β_6	$1_{\mathrm{Hol}(N)}$	$\rho(\eta^2)$	$\rho(\eta^3)\theta$	$\rho(\eta)\theta$

Conjugation by θ shows that β_{2i} gives the same structure as β_{2i-1}, $i = 1, 2, 3$.

Thus there are three Hopf-Galois structures on L/K along with the classical structure, which we call H_1, H_3 and H_5.

Searching for isomorphisms

Proposition

$H_i \cong H_j$ as K-Hopf algebras if and only if there exists a $\theta \in \mathrm{Aut}(N)$ such that

$$\overline{\beta_j}(x) = \theta\overline{\beta_i}(x)\theta^{-1} \text{ for all } x \in G.$$

	1_G	g	h	gh
β_1	$1_{\mathrm{Hol}(N)}$	$\rho(\eta)\theta$	$\rho(\eta^3)\theta$	$\rho(\eta^2)$
$\overline{\beta_1}$	$1_{\mathrm{Aut}(N)}$	θ	θ	$1_{\mathrm{Aut}(N)}$
β_3	$1_{\mathrm{Hol}(N)}$	$\rho(\eta)\theta$	$\rho(\eta^2)$	$\rho(\eta^3)\theta$
$\overline{\beta_3}$	$1_{\mathrm{Aut}(N)}$	θ	$1_{\mathrm{Aut}(N)}$	θ
β_5	$1_{\mathrm{Hol}(N)}$	$\rho(\eta^2)$	$\rho(\eta)\theta$	$\rho(\eta^3)\theta$
$\overline{\beta_5}$	$1_{\mathrm{Aut}(N)}$	$1_{\mathrm{Aut}(N)}$	θ	θ

Since $1_{\mathrm{Aut}(N)} \in Z(\mathrm{Aut}(N))$, such a $\theta \in \mathrm{Aut}(N)$ does not exist.
The three corresponding Hopf algebras are not isomorphic.

Still searching for isomorphisms

Corollary

Let $K \subseteq F \subseteq E,\ G_0 = \mathrm{Gal}(E/F)$ *Then* $F \otimes H_i \cong F \otimes H_j$ *as* K*-Hopf algebras if and only if there exists a* $\theta \in \mathrm{Aut}(N)$ *such that*

$$\overline{\beta_j}(g) = \theta \overline{\beta_i}(g) \theta^{-1} \text{ for all } g \in G_0.$$

	1_G	g	h	gh
$\overline{\beta_1}$	$1_{\mathrm{Aut}(N)}$	θ	θ	$1_{\mathrm{Aut}(N)}$
$\overline{\beta_3}$	$1_{\mathrm{Aut}(N)}$	θ	$1_{\mathrm{Aut}(N)}$	θ
$\overline{\beta_5}$	$1_{\mathrm{Aut}(N)}$	$1_{\mathrm{Aut}(N)}$	θ	θ

It is readily seen that

$$L^{\langle g \rangle} \otimes H_1 \cong L^{\langle g \rangle} \otimes H_3$$
$$L^{\langle h \rangle} \otimes H_1 \cong L^{\langle h \rangle} \otimes H_5$$
$$L^{\langle gh \rangle} \otimes H_3 \cong L^{\langle gh \rangle} \otimes H_5.$$

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$C_p \times C_p$: the structures

Let $p > 2$ be prime, and let L/K be Galois with $G = \mathrm{Gal}(L/K) \cong C_p \times C_p$.

The Hopf-Galois structures on L/K are known (Byott, 2002):

Let $t \in G$ be nontrivial, and let $T = \langle t \rangle$. Then $G = \langle s, t \rangle$ for some $s \in G$.

Pick $0 \leq d \leq p - 1$ and define

$$\alpha[s^k t^l] = s^k t^{l-1}$$
$$\beta[s^k t^l] = s^{k-1} t^{l+(k-1)d}.$$

Let $N_{T,d} = \langle \alpha, \beta \rangle$. Then $N_{T,d} \leq \mathrm{Perm}(G)$ is regular, and since

$${}^s\alpha = \alpha \qquad {}^t\alpha = \alpha \qquad {}^s\beta = \alpha^d\beta \qquad {}^t\beta = \beta$$

$N_{T,d}$ gives rise to a Hopf-Galois structure.

${}^s\alpha = \alpha,\ {}^t\alpha = \alpha,\ {}^s\beta = \alpha^d\beta,\ {}^t\beta = \beta$

Note that $g \in G$ acts trivially on $N_{T,d}, d \neq 0$ if and only if $g \in T$.

When are $N_{T_1,d_1} = \langle \alpha_1, \beta_1 \rangle$ and $N_{T_2,d_2} = \langle \alpha_2, \beta_2 \rangle$ isomorphic?

Assume $d_1 d_2 \neq 0$, i.e., the structures are nonclassical.

Suppose $\varphi : N_{T_1,d_1} \to N_{T_2,d_2}$ is G-equivariant.

Let $T_1 = \langle t_1 \rangle,\ \eta_1 \in N_{T_1,d_1}$. Then

$$ {}^{t_1}\varphi(\eta_1) = \varphi({}^{t_1}\eta_1) = \varphi(\eta_1) $$

so $\{\varphi(\eta_1) : \eta_1 \in N_{T_1,d_1}\} = N_{T_2,d_2}$ is fixed by T_1.

This can only happen if $T_1 = T_2$.

Now let $N_{T,d_1} = \langle \alpha_1, \beta_1 \rangle,\ N_{T,d_2} = \langle \alpha_2, \beta_2 \rangle, d_1 d_2 \neq 0$.

There exists a unique $1 \leq c \leq p-1$ such that $cd_2 \equiv d_1 \pmod{p}$.

Define $\varphi : N_{T,d_1} \to N_{T,d_2}$ by

$$\varphi(\alpha_1) = \alpha_2,\ \varphi(\beta_1) = \beta_2^c.$$

Since

$$\varphi({}^s\beta_1) = \varphi(\alpha_1^{d_1}\beta_1) = \alpha_2^{d_1}\beta_2^c = \alpha_2^{cd_2}\beta_2^c = {}^s\beta_2^c = {}^s\varphi(\beta_1),$$

the map φ is a G-equivariant isomorphism.

Summary

Let $H_{T,d}$ be the Hopf algebra corresponding to $N_{T,d},\ d > 0$.

- $H_{T_1,d_1} \ncong H_{T_2,d_2}$ if $T_1 \neq T_2$;
- $H_{T,d_1} \cong H_{T,d_2}$ for all $1 \leq d_1, d_2 \leq p - 1$;
- $H_{T,d} \ncong K[G]$.

This gives $(p + 1) + 1 = p + 2$ nonisomorphic Hopf algebras which provide at least one Hopf-Galois structure on L/K: the classical structure, and the structures given by $H_{T,1}$ for each of the $p + 1$ proper nontrivial subgroups T of G.

C_{p^n}: the structures

Let $p > 2$ be prime, and let L/K be Galois with $G = \mathrm{Gal}(L/K) = \langle g \rangle \cong C_{p^n}$.

Thanks to Kohl 1998, the Hopf-Galois structures are known, and most easily described using the holomorph. The only viable choice for N turns out to be cyclic as well, say $N = \langle \eta \rangle$.

Fix $\delta \in \mathrm{Aut}(N) \cong C_{p^{n-1}(p-1)}$ of order p^{n-1}.

Pick $0 \leq s < p^{n-1}$ and define $\beta_s : G \hookrightarrow \mathrm{Hol}(N)$ by

$$\beta_s(g) = \rho(g)\delta^s.$$

This provides p^{n-1} embeddings, all nonequivalent.

$\beta_s(g) = \rho(g)\delta^s$

Suppose $0 \leq r, s < p^{n-1}$ and let the Hopf algebras corresponding to β_r and β_s be H_r and H_s respectively.

Suppose $\theta \in \mathrm{Aut}(N)$ satisfies $\delta^s(g) = \overline{\beta_s}(g) = \theta\overline{\beta_r}(g)\theta^{-1}$.

Since $\mathrm{Aut}(N)$ is abelian, $\theta\overline{\beta_r}(g)\theta^{-1} = \overline{\beta_r}(g) = \delta^r(g)$.

Thus, $H_r \cong H_s$ if and only if $\delta^r = \delta^s$, i.e., $r = s$ since $|\delta| = p^{n-1}$.

$\beta_r(g) = \rho(g)\delta^r, \beta_s(g) = \rho(g)\delta^s$

Isomorphism via base change?

Let $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ be the unique maximal tower of field extensions.

Suppose $r \equiv s \pmod{p^{n-1-i}}$. Let $G_i = \langle g^{p^i} \rangle = \mathrm{Gal}(L/K_i)$

Then

$$\overline{\beta_r}(g^{p^i}) = \delta^{rp^i} = \delta^{sp^i} = \overline{\beta_s}(g^{p^i})$$

and hence $K_i \otimes H_r \cong K_i \otimes H_s$.

In fact, $K_i \otimes H_r \cong K_i \otimes H_s$ if and only if $r \equiv s \pmod{p)^{n-1-i}}$.

D_p: the structures

Let $p > 2$ be prime, and let L/K be Galois with $G = \mathrm{Gal}(L/K) = \langle r, s \rangle \cong D_p$ with $r^p = 1_G$.

The Hopf-Galois structures are known, thanks to Byott (2004).

Pick $0 \leq c \leq p-1$, and let $\eta_c = \lambda(r)\rho(r^c s) \in \mathrm{Perm}(D_p)$. Then $N_c := \langle \eta_c \rangle$ is a regular subgroup of $\mathrm{Perm}(D_p)$ normalized by D_p.

On each, the action of D_p on N_c remains the same:

$$^{r}\eta_c = \eta_c, \ {}^{s}\eta_c = \eta_c^{-1}.$$

The above, together with $\rho(D_p)$ and $\lambda(D_p)$, give the $p+1$ structures on L/K.

The corresponding Hopf algebras will be denoted H_c, $K[D_p]$, and H_λ.

Searching for isomorphisms

Let $K \subseteq F \subseteq L$. We have

- $K[D_p] \ncong H_\lambda, \; K[D_p] \ncong H_c$ from before.
- $F \otimes K[D_p] \cong F \otimes H_\lambda$ iff $F = E$ since D_p has trivial center.
- $F \otimes K[D_p] \ncong F \otimes H_c$ since H_c is a K-form of $L[N]$.
- $F \otimes H_\lambda \ncong F \otimes H_c$ since they are K-forms of different group rings.
- For $0 \leq c, d \leq p - 1$, $H_c \cong H_d$: the map $\eta_c \mapsto \eta_d$ is G-equivariant.

Thus, there are three distinct Hopf algebras which act on L/K, one of which acts in $p - 1$ different ways.

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Question

Given regular subgroups N_1, N_2 giving Hopf algebras H_1, H_2, when are $H_1 \cong H_2$ as K-algebras?

Unsolved (for now) at this level of generality.

Assumptions

1. N_1 and N_2 are abelian.
2. $\operatorname{char} K = 0$.

$E[N]$ is a separable K-algebra, subject to the classification from Monday's talk.

N abelian, char $K < 1$

Let H be the Hopf algebra corresponding to N.

Let $\mathcal{N}^D = \mathrm{Spec}(H)$. Then

$$\mathcal{N}^D(K^{\mathrm{sep}}) = \mathrm{Hom}_{K^{\mathrm{sep}}\text{-gp}}(\mathcal{N}_{K^{\mathrm{sep}}}, \mathbb{G}_{m,K^{\mathrm{sep}}}) \cong \mathrm{Hom}(N, (K^{\mathrm{sep}})^\times) = \widehat{N}.$$

Let $\Gamma = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Then Γ acts on $\widehat{N}$ by

$$(\gamma * \chi)[\eta] = \gamma(\chi[{}^{\gamma^{-1}}\eta])$$

Main result

Theorem (TARP, 2018)

Let $N_1, N_2 \leq Perm(X)$ be abelian regular subgroups normalized by G. Then $E[N_1]^G \cong E[N_2]^G$ as K-algebras iff there is a Γ-equivariant bijection $\widehat{N_1} \to \widehat{N_2}$.

Back to the biquadratic, looking for Γ-maps $\widehat{N_i} \to \widehat{N_j}$

Let us assume $\mathbb{Q}(i) \subseteq K$.
Here, $\widehat{N_1} = \widehat{N_3} = \widehat{N_5} = \langle \chi \rangle = C_4$, where $\chi(\eta) = i \in \mathbb{C}$. Γ acts through G. For each $z \in G$,

$$(z *_j \chi)[\eta] = z\chi[\,^z\eta] = z\chi[\,\overline{\beta_j}(z)(\eta)] = \begin{cases} i & \overline{\beta_j}(z) = 1_N \\ -i & \overline{\beta_j}(z) = \theta \end{cases}.$$

So $z *_j \chi = \chi$ if $\overline{\beta_j}(z) = 1_N$; otherwise, $z *_j \chi = \chi^3$.

	1_G	g	h	gh
$\overline{\beta_1}$	$1_{\mathrm{Aut}(N)}$	θ	θ	$1_{\mathrm{Aut}(N)}$
$\overline{\beta_3}$	$1_{\mathrm{Aut}(N)}$	θ	$1_{\mathrm{Aut}(N)}$	θ
$\overline{\beta_5}$	$1_{\mathrm{Aut}(N)}$	$1_{\mathrm{Aut}(N)}$	θ	θ

On $\widehat{N_1}$, χ is fixed by gh. Thus, if $H_3 \cong H_1$ then on $\widehat{N_3}$ we must have χ fixed by gh. Since this is not the case we conclude $H_1 \ncong H_3$.
Generalizing, one can show that H_1, H_3, and H_5 are all distinct as K-Hopf algebras.

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6 Looking forward

$G \cong C_p \times C_p$

We assume that K contains ζ, a primitive p^{th} root of unity.

Recall the non-classical Hopf algebras are of the form $H_{T,d}$ where $T = \langle t \rangle$ is a subgroup of $G = \langle s, t \rangle$ of order p and $1 \leq d \leq p-1$.

If $N = \langle \alpha, \beta \rangle \leq \text{Perm}(G)$ is the regular group corresponding to $H_{T,d}$, then $\widehat{N} = \langle \chi, \psi \rangle$ where

$$\chi(\alpha) = \zeta, \qquad \chi(\beta) = 1_N, \qquad \psi(\alpha) = 1_N, \qquad \psi(\beta) = \zeta$$

The action of G on $\widehat{N}$ is

$$s * \chi = \chi\psi^{p-d}, \qquad t * \chi = \chi, \qquad s * \psi = \psi, \qquad t * \psi = \psi.$$

Note that T acts trivially on $\widehat{N}$.

$s * \chi = \chi\psi^{p-d}, t * \chi = \chi, s * \psi = \psi, t * \psi = \psi$

Suppose $H_{T_1,d_1} \cong H_{T_2,d_2}$

By an argument similar to the Hopf algebra case, $T_1 = T_2$.

Of course, $H_{T,d_1} \cong H_{T,d_2}$ as K-algebras (since they are isomorphic as K-Hopf algebras).

Thus, the K-algebra isomorphism classes are the same as the K-Hopf algebra isomorphism classes.

In fact, $H_T \cong K^p \times (L^T)^{p-1}$ as K-algebras [Truman, 2016].

Cyclic, degree p^n

We assume a primitive $(p^n)^{\text{th}}$ root of unity ζ in K.

Recall: Hopf Galois structures $\leftrightarrow \beta_s : G \hookrightarrow \text{Hol}(N),\ \beta_s(g) = \rho(\eta)\delta^s$ with $|\delta| = p^{n-1} \in \text{Aut}(N)$.

TARP, 2018. For N cyclic, degree p^n, $\zeta \in K$, a Γ-equivariant bijection $\widehat{N} \to \widehat{N}$ exists if and only if a Γ-equivariant bijection $N \to N$ exists.

This result is not true if $\zeta \notin K$.

Since Γ factors through G, a G-equivariant bijection $\widehat{N} \to \widehat{N}$ exists if and only if a G-equivariant bijection $N \to N$ exists.

G acts via $\overline{\beta_s}$, so

$$^{g}\eta = \delta^s(\eta).$$

${}^g\eta = \delta^s(\eta)$.

Pick $0 < r, s \leq p^{n-1}$, and let v_p denote the p-adic valuation.

$v_p(r) = v_p(s)$. Then $\overline{\beta_r}(G) = \langle \delta^r \rangle = \langle \delta^s \rangle = \overline{\beta_s}(G)$. Thus the orbits of any $\eta \in N$ are the same with respect to either action, allowing for a G-equivariant bijection.

$v_p(r) < v_p(s)$. Then $\overline{\beta_r}(G) = \langle \delta^r \rangle \supsetneq \langle \delta^s \rangle = \overline{\beta_s}(G)$ so the orbits do not coincide.

Thus, the Hopf algebras given by β_r, β_s are isomorphic as K-algebras if and only if $v_p(r) = v_p(s)$.

$H_r \cong H_s \Leftrightarrow v_p(r) = v_p(s)$

Also, [TARP, 2018] shows

$$H_r \cong K^{p^{1+v_p(r)}} \times \prod_{m=1}^{n-1-v_p(r)} (K_m)^{p^{v_p(r)}(p-1)},$$

where $K = K_0 \subset K_1 \subset \cdots \subset K_n = L$ as before.

In [Childs, 2011], Lindsay obtains an explicit set of K-algebra generators for each H_s, from which one could obtain the same results.

The D_p case

Recall that the regular subgroups are $\rho(D_p), \ \lambda(D_p)$, and the collection $\{N_c : 0 \leq c \leq p-1\}$.

We denote the Hopf algebras $K[D_p], H_\lambda, H_0, H_1, \dots, H_{p-1}$.

We know that $H_c \cong H_d$ as K(-Hopf) algebras.

Since H_c is commutative, it cannot be isomorphic to the non-commutative algebras $K[D_p]$ or H_λ, even after base change.

This leaves the question of whether $K[D_p] \cong H_\lambda$, a case not covered by the work above.

However, Greither has announced a proof of a very general result:

Theorem

Let L/K be Galois, group G nonabelian. Let H_λ be the Hopf algebra which provides the canonical nonclassical Hopf-Galois structure. Then $H_\lambda \cong K[G]$ as K-algebras.

This, of course, implies that any Hopf algebra arising from a fixed-point free endomorphism is also isomorphic to $K[G]$ as a K-algebra.

Outline

1. Statement of the problems
2. Hopf algebra isomorphism problems
3. Examples
   - Elementary abelian degree p^2
   - Cyclic, degree p^n
   - Dihedral, degree $2p$
4. Algebra isomorphism problems
5. Examples
   - Elementary abelian degree p^2
   - Cyclic, degree p^n
   - Dihedral, degree $2p$
6. Looking forward

Mission accomplished?

If:

- $\mathrm{char}(K) = 0$,
- N is abelian, preferably cyclic,
- $\zeta_{|N|} \in K$,

then we have a pretty good idea of what's going on with the algebra structure.

fait a-incompli

If:

- $\mathrm{char}(K) = 0$,
  Can be easily extended to $\gcd(\mathrm{char}(K), [E : K]) = 1$.
- N is abelian, preferably cyclic,
  Might be able to use the ideas from yesterday's talk to replace $\widehat{N}$ with N in certain circumstances (e.g., if we can show the Hopf algebras are not isomorphic but their underlying coalgebras are, perhaps we can conclude they are not isomorphic as algebras. (Big Might.))
- $\zeta_{|N|} \in K$,
  It is possible that yesterday's talk might help here as well since Γ acts on $\mathrm{Spec}(H_i{}^*)(E)$ through G.

then we'll have a pretty good idea of what's going on with the algebra structure.

Thank you.